

On the de Rham-Wu decomposition for Riemannian and Lorentzian manifolds

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ABSTRACT. It is explained how to find the de Rham decomposition of a Riemannian manifold and the Wu decomposition of a Lorentzian manifold. For that it is enough to find parallel symmetric bilinear forms on the manifold, and do some linear algebra. This result will allow to compute the connected holonomy group of an arbitrary Riemannian or Lorentzian manifold.

Keywords: Lorentzian manifold, holonomy group, de Rham decomposition, Wu decomposition

1. Introduction

The classification of connected holonomy groups of indecomposable Riemannian manifolds is a classical result that has many applications both in geometry and theoretical physics, in particular, in string theory compactifications and M-theory, see [3, 6, 9, 12, 15, 16] and references therein.

The classification of connected holonomy groups of indecomposable Lorentzian manifolds is available as well [7, 13]. Holonomy groups of 4-dimensional Lorentzian manifolds and their relation to General Relativity are studied e.g. in [13, 11]. Recently an attention to the holonomy groups of Lorentzian manifolds of arbitrary dimension is paid in the physical literature [4, 5, 6, 8, 10].

It is important to find the connected holonomy group of an arbitrary Riemannian or Lorentzian manifold. De Rham and Wu theorems allow to decompose (at least locally) a Riemannian or Lorentzian manifold into a product of indecomposable manifolds. This implies that the holonomy group of the manifold is a product of the holonomy groups of indecomposable manifolds. Thus in order to find the holonomy group of an arbitrary Riemannian or Lorentzian manifold one should first to find its de Rham or Wu decomposition. The proof of the de Rham and Wu decomposition theorems assumes that the holonomy group is known, so a priori it is unclear how to find these decompositions without knowing the holonomy group.

In this paper we give algorithms that allow to find the de Rham decomposition of a Riemannian manifold and the Wu decomposition of a Lorentzian manifold. In order to find these decompositions, it is enough to find parallel symmetric bilinear forms on the manifold, and then do some linear algebra in the tangent space at a point of the manifold. Consequently, the algorithms can be computerized, e.g. as a part of the package DifferentialGeometry for Maple [2].

In [3], it is explained how to find the connected holonomy group of an indecomposable Riemannian manifold, for that one can analyze parallel differential forms on the manifold. In another paper we will explain how to find the holonomy group of an indecomposable Lorentzian manifold. This and the results of the present paper will provide the complete algorithm that allows to find the connected holonomy group of an arbitrary Riemannian or Lorentzian manifold.

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2. Holonomy groups

The theory of holonomy groups of pseudo-Riemannian manifolds can be found e.g. in [3, 12].

Let (M, g) be a connected pseudo-Riemannian manifold of signature (r, s) (r is the number of minuses in the signature of the metric g). We will be interested in the case of Riemannian manifolds ($r = 0$, i.e. g is positive definite) and in the case of Lorentzian manifolds ($r = 1$).

Denote by ∇ the Levi-Civita connection on M defined by the metric g ; ∇ is the unique torsion-free linear connection on M that preserves the metric g : $\nabla g = 0$. Let $\gamma : [a, b] \subset \mathbb{R} \rightarrow M$ be a piecewise smooth curve on M . The connection ∇ defines the parallel transport $\tau_\gamma : T_{\gamma(a)}M \rightarrow T_{\gamma(b)}M$, which is an isomorphism of the pseudo-Euclidean spaces $(T_{\gamma(a)}M, g_{\gamma(a)})$ and $(T_{\gamma(b)}M, g_{\gamma(b)})$.

The *holonomy group* G_x of (M, g) at a point $x \in M$ is the Lie group that consists of the pseudo-orthogonal transformations given by the parallel transports along all piecewise smooth loops at the point x . It can be identified with a Lie subgroup of the pseudo-orthogonal Lie group $O(r, s) = O(T_x M, g_x)$.

Recall that a tensor field T on (M, g) is *parallel* if $\nabla T = 0$, or equivalently T is preserved by parallel transports: for any piecewise smooth curve starting at a point $y \in M$ with an end-point $z \in M$ it holds $\tau_\gamma T_y = T_z$, where τ_γ is the extension of the parallel transport along γ to tensors.

The *fundamental principle for the holonomy groups* [3] states that there exists a one-to-one correspondents between parallel tensor fields T on M and tensors T_0 of the same type at x preserved by the tensor extension of the representation of the holonomy group.

3. The de Rham decomposition for Riemannian manifolds

In this section we will give an algorithm allowing to decompose (at least locally) any Riemannian manifold with not irreducible holonomy group in the product of a flat Riemannian manifold and of Riemannian manifolds with irreducible holonomy groups. This decomposition allows to restrict attention to Riemannian manifolds with irreducible holonomy groups.

We consider a Riemannian manifold (M, g) of dimension n with the holonomy group $G \subset O(n)$ at a point $x \in M$.

If (N, h) is another Riemannian manifold of dimension m with the holonomy group $H \subset O(m)$ at a point $y \in N$, then the product $(M \times N, g + h)$ is a Riemannian manifold with the holonomy group $G \times H \subset O(n+m)$ at the point $(x, y) \in M \times N$; this holonomy group preserves the subspaces $T_x M, T_y N \subset T_{(x,y)}(M \times N)$. This statement can be inverted in the following way.

The *de Rham decomposition Theorem* [3] states that if (M, g) is simply connected and complete, and its holonomy group is not irreducible, then (M, g) can be decomposed into the product of a flat Riemannian manifold (M_0, g_0) and of Riemannian manifolds $(M_1, g_1), \dots, (M_r, g_r)$ with irreducible holonomy groups. For general (M, g) with a not irreducible connected holonomy group such decomposition exists only locally. Thus the irreducibility of the connected holonomy group of a Riemannian manifold (M, g) is equivalent to the local indecomposability of (M, g) .

Let us explain where the de Rham decomposition comes from. Let $x \in M$. Since the holonomy group $G \subset O(n)$ is totally reducible, the tangent space $T_x M$ can be decomposed into an orthogonal direct sum

$$(1) \quad T_x M = E_{0x} \oplus E_{1x} \oplus \dots \oplus E_{rx},$$

where E_{0x} is the subspace consisting of G -invariant vectors, each subspace $E_{\alpha x} \subset T_x M$, $1 \leq \alpha \leq r$, is G -invariant and the induced representation is irreducible.

The subspaces $E_{0x}, \dots, E_{rx} \subset T_x M$ define by means of the parallel transport parallel distributions E_0, \dots, E_r on M . These distributions are involutive, and the manifolds M_0, \dots, M_r from the above decomposition are maximal integral manifolds of these distributions passing through the point x . The metrics g_α are the restrictions of g to these distributions.

The holonomy group of (M, g) is the product

$$G = G_1 \times \dots \times G_r,$$

where G_α is the restriction of G to $E_{\alpha x}$. If the decomposition of (M, g) is global, then G_α is the holonomy group of the manifold (M_α, g_α) . However, if the decomposition is not global, then the holonomy group of (M_α, g_α) may be a proper subgroup of G_α ; in that case G_α is the holonomy group of the induced connection on the distribution E_α considered as a vector bundle over M .

The task is to find the distributions E_α . We may find E_0 as the distribution consisting of all parallel vector fields. Then we may work with E_0^\perp and g restricted to it. This allows us to assume that $E_0 = 0$.

Note that if (M, g) is indecomposable, then the dimension of parallel symmetric bilinear forms on (M, g) equals to one. This follows from the Fundamental principle for holonomy groups and from the fact that any element in the second symmetric power $\odot^2 \mathbb{R}^n$ of $T_x M \simeq \mathbb{R}^n$ preserved by the irreducible subgroup $G \subset O(n)$ is proportional to the metric g_x at the point x . In general, the dimension of parallel symmetric bilinear forms on (M, g) equals to r (we assume that $E_0 = 0$) and this real vector space is generated by g_1, \dots, g_r ; here we assume that $g_\alpha|_{E_\alpha \times E_\alpha} = g|_{E_\alpha \times E_\alpha}$ and $g_\alpha|_{E_\beta \times E} = 0$ if $\alpha \neq \beta$. In other words,

$$(2) \quad g_\alpha(\cdot, \cdot) = g(\text{pr}_{E_\alpha} \cdot, \text{pr}_{E_\alpha} \cdot),$$

where the projection is taken with respect to decomposition (1).

Finding all parallel symmetric bilinear forms on (M, g) (e.g. with Maple), we get an answer in the form

$$c_1 \tilde{g}_1 + \dots + c_r \tilde{g}_r,$$

where $c_1, \dots, c_r \in \mathbb{R}$ are arbitrary and $\tilde{g}_1, \dots, \tilde{g}_r$ is a basis of the space of all parallel symmetric bilinear forms on (M, g) . Since g is parallel, we may assume that $\tilde{g}_1 = g$ (indeed, we may find a linear independent subsystem in $\{g, \tilde{g}_1, \dots, \tilde{g}_r\}$ that contains g).

We may write

$$(3) \quad \tilde{g}_\alpha = \sum_{\beta=1}^r A_{\beta\alpha} g_\beta, \quad A_{\beta\alpha} \in \mathbb{R}.$$

Since we need to find the numbers $A_{\beta\alpha} \in \mathbb{R}$, we may work with a fixed point $x \in M$. The proof of the following proposition will allow to find the matrix $(A_{\beta\alpha})$ using some linear algebra.

PROPOSITION 1. *Let V be a vector space with the Euclidean metric η . Suppose that an orthogonal decomposition*

$$(4) \quad V = V_1 \oplus \dots \oplus V_r$$

is fixed. Let

$$\eta_\alpha(\cdot, \cdot) = \eta(\text{pr}_{V_\alpha} \cdot, \text{pr}_{V_\alpha} \cdot), \quad \alpha = 1, \dots, r.$$

If an arbitrary basis $\tilde{\eta}_1, \dots, \tilde{\eta}_r$ of the vector space $\text{span}\{\eta_1, \dots, \eta_r\}$ is given, then the forms η_α and the decomposition (4) can be reconstructed up to a permutation.

Proof. We obtain the relation

$$\tilde{\eta}_\alpha = \sum_{\beta=1}^r A_{\beta\alpha} \eta_\beta, \quad A_{\beta\alpha} \in \mathbb{R}, \quad \alpha = 1, \dots, r.$$

We may assume that $\tilde{\eta}_1 = \eta$. Consider $\tilde{\eta}_2$, then

$$V = F \oplus F^\perp,$$

where

$$F = \ker \tilde{\eta}_2 = \{X \in V | \tilde{\eta}_2(X, Y) = 0 \text{ for all } Y \in V\}$$

is the kernel of $\tilde{\eta}_2$ and F^\perp is its orthogonal complement with respect to η . Both F and F^\perp consist of some of V_α , i.e. this decomposition is orthogonal with respect to all tensors η_α . Consider the decomposition $V = F \oplus F^\perp$, take the restrictions of $\tilde{\eta}_3$ to each of these spaces and decompose F and F^\perp in the same manner. Continue this process for all $\tilde{\eta}_\alpha$, then we get a decomposition

$$(5) \quad V = F_1 \oplus \dots \oplus F_s$$

such that the restriction of each $\tilde{\eta}_\alpha$ to any of F_k is either zero or non-degenerate. Now we continue to subdivide this decomposition. Let α run from 2 to r . Consider the restrictions $\tilde{\eta}_\alpha|_{F_k \times F_k}$ and $\eta|_{F_k \times F_k}$ of $\tilde{\eta}_\alpha$ and η to each F_k . If the restriction $\tilde{\eta}_\alpha|_{F_k \times F_k}$ is non-zero (i.e. $\tilde{\eta}_\alpha|_{F_k \times F_k}$ is non-degenerate), and $\tilde{\eta}_\alpha|_{F_k \times F_k}$ is not proportional to $\eta|_{F_k \times F_k}$, then we consider a new $\tilde{\eta}_\alpha$: its restriction to $F_l \times F_l$ for $l \neq k$ remains the same, and we change its restriction $\tilde{\eta}_\alpha|_{F_k \times F_k}$ to

$\tilde{\eta}_\alpha|_{F_k \times F_k} - b\eta|_{F_k \times F_k}$, where $b \in \mathbb{R}$ is a number such that the restriction $(\tilde{\eta}_\alpha - b\eta)|_{F_k \times F_k}$ to F_k is degenerate. To find such b , take any vector $X \in F_k$ such that $\eta(X, X), \tilde{\eta}_\alpha(X, X) \neq 0$ (in the case of positive definite η , any non-zero X satisfies this condition) and set $b = \frac{\tilde{\eta}_\alpha(X, X)}{\eta(X, X)}$. Using the new tensor $\tilde{\eta}_\alpha$, we may subdivide F_k (since now $\tilde{\eta}_\alpha|_{F_k \times F_k}$ is degenerate and non-zero), i.e. we subdivide the decomposition (5). Continue this process. At the end we will get that the restriction of any $\tilde{\eta}_\alpha$ to any F_k is either zero, or it is proportional to $\eta|_{F_k \times F_k}$. This means that the number s in decomposition (5) equals r , that is, decomposition (5) is the decomposition (4), $F_\alpha = V_\alpha$ (up to a renumbering). Then we may find the forms η_α , and using the initial forms $\tilde{\eta}_\alpha$, we find the matrix $(A_{\beta\alpha})$. Finally, $V_\alpha = (\ker \eta_\alpha)^\perp$. This proves the proposition. \square

Thus using (3) considered at the point x , we find the matrix $(A_{\beta\alpha})$. Then using (3) and the inverse matrix, we find the metrics g_α . Now for any $y \in M$ find $E_{\alpha y}$ using one of the formulas:

$$(6) \quad E_{\alpha y} = (\ker g_{\alpha y})^\perp, \quad \text{or}$$

$$(7) \quad E_{\alpha y} = \cap_{\beta \neq \alpha} \ker g_{\beta y} = \{X \in T_y M | g_{\beta y}(X, \cdot) = 0 \text{ for all } \beta \neq \alpha\}.$$

Thus we know the distributions E_α .

3.1. Example. In order to find a decomposable metric, we simply take the local metric on the product of two spheres:

$$g = (dy_1)^2 + \sin^2 y_1 (dy_2)^2 + (dy_3)^2 + \sin^2 y_3 (dy_4)^2, \quad 0 \leq y_1, y_3 \leq \pi.$$

Consider the new coordinates

$$x_1 = y_1 - y_3, \quad x_2 = y_2 - y_4, \quad x_3 = y_3, \quad x_4 = y_4.$$

The metric takes the form

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \sin^2(x_1 + x_3) & 0 & \sin^2(x_1 + x_3) \\ 1 & 0 & 2 & 0 \\ 0 & \sin^2(x_1 + x_3) & 0 & \sin^2 x_3 + \sin^2(x_1 + x_3) \end{pmatrix}.$$

Looking at this metric, it is not obvious that it is decomposable. Now we will apply the above algorithm in order to decompose the obtained metric. Using Maple, we get that there are no non-zero parallel vector fields and parallel symmetric bilinear forms are the following:

$$\begin{pmatrix} C_2 & 0 & C_2 & 0 \\ 0 & C_2 \sin^2(x_1 + x_3) & 0 & C_2 \sin^2(x_1 + x_3) \\ C_2 & 0 & C_1 & 0 \\ 0 & C_2 \sin^2(x_1 + x_3) & 0 & C_1 \sin^2 x_3 + C_2(\sin^2(x_1 + x_3) - \sin^2 x_3) \end{pmatrix}.$$

Let $\tilde{g}_1 = g$, $\tilde{g}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2 x_3 \end{pmatrix}$. Consider the point $x = (0, 0, \frac{\pi}{2}, 0)$. We obtain the

decomposition

$$T_x M = F \oplus F^\perp = \ker \tilde{g}_{2x} \oplus (\ker \tilde{g}_{2x})^\perp,$$

$$\ker \tilde{g}_{2x} = \text{span}\{(\partial_{x_1})_x, (\partial_{x_2})_x\}, \quad (\ker \tilde{g}_{2x})^\perp = \text{span}\{(\partial_{x_1} - \partial_{x_3})_x, (\partial_{x_2} - \partial_{x_4})_x\}.$$

Since the number of the summands in this decomposition equals to the dimension of parallel symmetric bilinear forms, we get

$$E_{1x} = \ker \tilde{g}_{2x}, \quad E_{2x} = (\ker \tilde{g}_{2x})^\perp.$$

Using (2) evaluated at x , we obtain

$$g_{1x} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad g_{2x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conclude that $g_x = g_{1x} + g_{2x}$, $\tilde{g}_{2x} = g_{2x}$, and

$$g_1 = g - \tilde{g}_2, \quad g_2 = \tilde{g}_2.$$

Using (7), we get that the distribution E_1 is spanned by the vector fields ∂_{x_1} , ∂_{x_2} , and E_2 is spanned by the vector fields $\partial_{x_1} - \partial_{x_3}$, $\partial_{x_2} - \partial_{x_4}$. We find the coordinates z_1, \dots, z_4 adopted to the decomposition $TM = E_1 \oplus E_2$ requiring that

$$\partial_{z_1} = \partial_{x_1}, \quad \partial_{z_2} = \partial_{x_2}, \quad \partial_{z_3} = \partial_{x_1} - \partial_{x_3}, \quad \partial_{z_4} = \partial_{x_2} - \partial_{x_4},$$

and we get

$$z_1 = x_1 + x_3, \quad z_2 = x_2 + x_4, \quad z_3 = -x_3, \quad z_4 = -x_4.$$

With respect to these coordinates it holds

$$g = (dz_1)^2 + \sin^2 z_1 (dz_2)^2 + (dz_3)^2 + \sin^2 z_3 (dz_4)^2,$$

i.e. we obtain the initial metric.

4. The Wu decomposition for Lorentzian manifolds

The *Wu decomposition Theorem* [17] generalizes the de Rham Theorem for the case of pseudo-Riemannian manifolds. It states that a pseudo-Riemannian manifold (M, g) with not weakly irreducible connected holonomy group can be decomposed at list locally into the product of pseudo-Riemannian manifolds $(M_0, g_0), \dots, (M_r, g_r)$ such that (M_0, g_0) is flat and the holonomy groups of $(M_1, g_1), \dots, (M_r, g_r)$ are weakly irreducible. Recall that a subgroup of a pseudo-orthogonal group is weakly irreducible, if it does not preserve any proper non-degenerate subspace of the pseudo-Euclidean space. Consequently, a pseudo-Riemannian manifold (M, g) is locally indecomposable if and only if its connected holonomy group is weakly irreducible. In that case E_0 is not the distribution that consists of all parallel vector fields on (M, g) , but it is a non-degenerate subdistribution of the last one and in general it is not defined uniquely. Note that if the holonomy group is weakly irreducible and not irreducible, then it preserves an isotropic subspace of the tangent space, but this does not imply the local decomposability of the manifold.

The algorithm of Section 3 works also for pseudo-Riemannian manifolds if we know that the holonomy group of each factor in the decomposition is irreducible. In that case the dimension of parallel symmetric bilinear forms equals to the number of the manifolds in the decomposition (without loss of generality we assume that $E_0 = 0$), then (3) holds. The problem is that a locally indecomposable pseudo-Riemannian manifold may admit parallel symmetric bilinear forms not proportional to the metric. The structure of such forms is known only in the case of Lorentzian manifolds.

Now we consider a Lorentzian manifold (M, g) . And we will obtain the Wu decomposition of that manifold.

A.V. Aminova [1] proved that if a locally indecomposable Lorentzian manifold (M, g) admits a parallel symmetric bilinear form not proportional to the metric, then (M, g) admits a parallel light-like vector field p , the space of parallel bilinear forms is 2-dimensional, and it is spanned by g and $\theta \otimes \theta$, where $\theta = g(p, \cdot)$ is the 1-form corresponding to p (see also the paper [14] by G.S. Hall for the case of dimension 4).

It is clear that in the Wu decomposition of a Lorentzian manifold only one submanifold is Lorentzian, and all the other are Riemannian. The Lorentzian part is locally indecomposable and admits a parallel light-like vector field if and only if the restriction of g to the space of parallel vector fields is degenerate (this property may be checked at a single point if we restrict the parallel vector fields to that point). If the Lorentzian part is contained in the distribution E_0 , or it does not admit a parallel light-like vector field, then the algorithm of Section 3 works.

Suppose that the Lorentzian part (that we assume to be (M_r, g_r)) in the Wu decomposition is indecomposable and admits a parallel vector field p . Let $\tilde{E}_0 \subset TM$ be the subbundle spanned by all parallel vector fields on (M, g) . Then, $p \in \Gamma(\tilde{E}_0)$. Let $E_{0x} \subset \tilde{E}_{0x}$ be any subspace complementary to $\mathbb{R}p_x$. Let $E_0 \subset \tilde{E}_0$ be the subbundle spanned by parallel vector fields with values in E_{0x} at the point x . Then $E_0 \subset TM$ is a parallel subbundle and the restriction of g to E_0 is non-degenerate. We consider E_0^\perp and the restriction of g to it. Hence we again may assume that $E_0 = 0$. Then

the space of parallel bilinear forms on (M, g) is spanned by $g_1, \dots, g_r, \theta \otimes \theta$ and it is of dimension $r + 1$.

Finding all parallel symmetric bilinear forms on (M, g) , we get an answer in the form

$$c_1 \bar{g}_1 + \dots + c_{r+1} \bar{g}_{r+1},$$

where $c_1, \dots, c_{r+1} \in \mathbb{R}$ are arbitrary and $\bar{g}_1, \dots, \bar{g}_{r+1}$ is a basis of the space of all parallel symmetric bilinear forms on (M, g) . Since g is parallel, we may assume that $\bar{g}_1 = g$.

There exist real numbers $(C_{\beta\alpha})_{\beta,\alpha=1}^{r+1}$ such that

$$(8) \quad \bar{g}_\alpha = \sum_{\beta=1}^r C_{\beta\alpha} g_\beta + C_{r+1\alpha} \theta \otimes \theta.$$

Again, we may find the matrix $C_{\beta\alpha}$ sitting at a single point and doing linear algebra.

PROPOSITION 2. *Let V be a vector space with the Minkowski metric η . Suppose that an orthogonal decomposition*

$$(9) \quad V = V_1 \oplus \dots \oplus V_r$$

is fixed such that the restriction of η to V_r is of Minkowski signature. Let $p \in V_r$ be a fixed non-zero isotropic vector, $\theta = g(p, \cdot)$, and let

$$\eta_\alpha(\cdot, \cdot) = \eta(\text{pr}_{V_\alpha} \cdot, \text{pr}_{V_\alpha} \cdot), \quad \alpha = 1, \dots, r.$$

If an arbitrary basis $\bar{\eta}_1, \dots, \bar{\eta}_{r+1}$ of the vector space $\text{span}\{\eta_1, \dots, \eta_r, \theta \otimes \theta\}$ is given, then the forms η_α and the decomposition (9) can be reconstructed up to a permutation of the subspaces V_1, \dots, V_{r-1} .

Proof. Again we may assume that $\bar{\eta}_1 = \eta$. Consider the relation

$$(10) \quad \bar{\eta}_\alpha = \sum_{\beta=1}^r C_{\beta\alpha} \eta_\beta + C_{r+1\alpha} \theta \otimes \theta.$$

In particular, $(C_{\beta 1})_{\beta=1}^{r+1} = (1, 0, \dots, 0)$.

Let $q \in V$ be a light-like vector not proportional to p , i.e. $\theta(q) = \eta(p, q) \neq 0$. To find such vector it is enough to take any vector $X \in V$ such that $\eta(p, X) \neq 0$, and if $\eta(X, X) \neq 0$, then take $q = p - \frac{2\eta(p, X)}{\eta(X, X)} X$. It is clear that

$$\bar{\eta}_\alpha(p, q) = C_{r\alpha} \eta_r(p, q) = C_{r\alpha} \eta(p, q), \quad 2 \leq \alpha \leq r + 1.$$

This allows to find the coefficients $C_{r\alpha}$. Changing each $\bar{\eta}_\alpha$ to $\bar{\eta}_\alpha - C_{r\alpha} \eta$, we obtain that $C_{r\alpha} = 0$ for $2 \leq \alpha \leq r + 1$.

The following three lemmas will allow to get the algorithm.

LEMMA 1. *Let $2 \leq \alpha \leq r + 1$. If $C_{r+1\alpha} = 0$, then*

$$\ker \bar{\eta}_\alpha = \oplus_{1 \leq \beta \leq r, C_{\beta\alpha}=0} V_\beta,$$

and

$$\ker \eta|_{\ker \bar{\eta}_\alpha \times \ker \bar{\eta}_\alpha} = 0.$$

If $C_{r+1\alpha} \neq 0$, then

$$\ker \bar{\eta}_\alpha = \oplus_{1 \leq \beta \leq r-1, C_{\beta\alpha}=0} V_\beta \oplus \{X \in V_r | \theta(X) = 0\},$$

and

$$\ker \eta|_{\ker \bar{\eta}_\alpha \times \ker \bar{\eta}_\alpha} = \mathbb{R}p.$$

Proof. Suppose that $C_{r+1\alpha} = 0$. If $C_{\beta\alpha} = 0$ then it is clear that $V_\beta \subset \ker \bar{\eta}_\alpha$. Let $X \in \ker \bar{\eta}_\alpha$. We may write $X = X_1 + \dots + X_r$, where $X_\gamma \in V_\gamma$. Suppose that $C_{\beta\alpha} \neq 0$. Let $Y \in V_\beta$ be any vector. Then, $0 = \bar{\eta}_\alpha(X, Y) = C_{\beta\alpha} \eta_\beta(X_\beta, Y)$ for any $Y \in V_\beta$, i.e. $X_\beta = 0$. This implies the first equality of the first statement. The second equality is obvious, since η is non-degenerate on each V_β .

Suppose that $C_{r+1\alpha} \neq 0$. The inclusion \supset in the first equality is obvious. Let $X \in \ker \bar{\eta}_\alpha$. We write $X = X_1 + \dots + X_r$, where $X_\gamma \in V_\gamma$. Suppose that $C_{\beta\alpha} \neq 0$, $1 \leq \beta \leq r - 1$. As

above, this implies $X_\beta = 0$. Let $Y \in V_r$ be a vector such that $\theta(Y) \neq 0$. Then $0 = \bar{\eta}_\alpha(X, Y) = C_{r+1\alpha}\theta(X_r)\theta(Y)$, i.e. $\theta(X_r) = 0$. This proves the second statement. \square

This lemma allows us easily indicate whether $C_{r+1\alpha} = 0$ or not: we should compute $\ker \bar{\eta}_\alpha$ and $\ker \eta|_{\ker \bar{\eta}_\alpha \times \ker \bar{\eta}_\alpha}$. If $C_{r+1\alpha} = 0$ for some $\alpha \geq 2$, then we add $\theta \otimes \theta$ to $\bar{\eta}_\alpha$. Then by the lemma we get

$$\ker \bar{\eta}_\alpha = \oplus_{1 \leq \beta \leq r-1, C_{\beta\alpha}=0} V_\beta \oplus \{X \in V_r | \theta(X) = 0\}, \quad 2 \leq \alpha \leq r+1.$$

Let us consider the following vector space:

$$W = \cap_{\alpha=2}^{r+1} \ker \bar{\eta}_\alpha.$$

LEMMA 2. *It holds*

$$W = \{X \in V_r | \theta(X) = 0\},$$

$$W^\perp_\eta = V_1 \oplus \dots \oplus V_{r-1} \oplus \mathbb{R}p.$$

Proof. We claim that for any β , $1 \leq \beta \leq r-1$ there exists an α , $2 \leq \alpha \leq r+1$, such that $C_{\beta\alpha} \neq 0$. Indeed, if the claim is wrong then there exists a β , $1 \leq \beta \leq r-1$ such that it holds

$$(C_{\beta\alpha})_{\alpha=1}^{r+1} = (C_{r\alpha})_{\alpha=1}^{r+1} = (1, 0, \dots, 0),$$

i.e. the matrix $(C_{\beta\alpha})_{\beta, \alpha=1}^{r+1}$ is degenerate, that gives a contradiction. The first equality follows from the claim. The second equality is obvious. \square

LEMMA 3. *The intersection $\cap_{\alpha=2}^{r+1} (W^\perp_\eta)^{\perp_{\bar{\eta}_\alpha}}$ that can be written as*

$$\{X \in V | \bar{\eta}_\alpha(X, Y) = 0 \text{ for all } 2 \leq \alpha \leq r+1, Y \in W^\perp_\eta\}$$

coincides with V_r .

Proof. Suppose that $X \in \cap_{\alpha=2}^{r+1} (W^\perp_\eta)^{\perp_{\bar{\eta}_\alpha}}$. Then for any α , $2 \leq \alpha \leq r+1$ and all $Y \in W^\perp_\eta$ it holds $\bar{\eta}_\alpha(X, Y) = 0$. Consider the decomposition $X = X_1 + \dots + X_r$. Let $1 \leq \beta \leq r-1$. Above we have seen that there exist α , $2 \leq \alpha \leq r+1$ such that $C_{\beta\alpha} \neq 0$. Let $Y \in V_\beta \subset W^\perp_\eta$. Then,

$$0 = \eta_\alpha(X, Y) = C_{\beta\alpha}\eta_\beta(X, Y) = C_{\beta\alpha}\eta_\beta(X_\beta, Y).$$

Consequently, $X_\beta = 0$, and $X = X_r \in V_r$.

Conversely, if $X \in V_r$, $2 \leq \alpha \leq r+1$, $Y \in W^\perp_\eta$, then $\eta_\alpha(X, Y) = C_{r+1\alpha}\theta(X)\theta(Y) = 0$, since $\theta(Y) = 0$. This proves the lemma. \square

Thus in order to find the space V_r , it is enough to compute the spaces

$$W = \cap_{\alpha=2}^{r+1} \ker \bar{\eta}_\alpha, \quad W^\perp_\eta, \quad \cap_{\alpha=2}^{r+1} (W^\perp_\eta)^{\perp_{\bar{\eta}_\alpha}}.$$

Now we consider $(V_r)^\perp_\eta$ and the restrictions of the forms $\bar{\eta}_1 = \eta, \bar{\eta}_2, \dots, \bar{\eta}_{r+1}$ to $(V_r)^\perp_\eta$. Clearly, the rank of this system equals to $r-1$. Let $\tilde{\eta}_1, \dots, \tilde{\eta}_{r-1}$ be a linearly independent subsystem such that $\tilde{\eta}_1 = \eta$. There exist real numbers $(A_{\beta\alpha})_{\beta, \alpha=1}^{r-1}$ such that

$$(11) \quad \tilde{\eta}_\alpha = \sum_{\beta=1}^{r-1} A_{\beta\alpha}\eta_\beta.$$

These numbers can be found in the same way as in Section 3. Using (11) and the inverse matrix to $A_{\beta\alpha}$, we find the forms $\eta_1, \dots, \eta_{r-1}$. Next, $\eta_r = \eta - \eta_1 - \dots - \eta_{r-1}$ and $\theta = \eta(p, \cdot) = \eta_r(p, \cdot)$. Using the initial $\bar{\eta}_1, \dots, \bar{\eta}_{r+1}$ and (10), we find the matrix $(C_{\beta\alpha})$. Now, $V_\alpha = (\ker \eta_\alpha)^\perp_\eta$. The proposition is proved. \square

Thus we have found the matrix $(C_{\beta\alpha})$ from the equation (8). Then using (8) and the inverse matrix to $(C_{\beta\alpha})$, we find the metrics g_α . Now the distributions V_α can be found using (6) or (7).

4.1. Example. Consider the following product metric

$$g = (dy_1)^2 + \sin^2 y_1 (dy_2)^2 + 2dy_3 dy_5 + (dy_4)^2 + y_4^2 (dy_5)^2.$$

After the coordinate transformation

$$x_1 = y_1 - y_4, \quad x_2 = y_2, \quad \dots, \quad x_5 = y_5$$

the metric takes the form

$$g = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & \sin^2(x_1 + x_4) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & x_4^2 \end{pmatrix}.$$

Now we apply the algorithm of this section to the obtained metric. The space of parallel vector fields is one-dimensional and it is spanned by the light-like vector field $p = \partial_{x_3}$. Hence, $\theta = dx_5$. The space of parallel bilinear symmetric forms is three-dimensional and consists of the following matrices:

$$\begin{pmatrix} C_1 & 0 & 0 & C_1 & 0 \\ 0 & C_1 \sin^2(x_1 + x_4) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_3 \\ C_1 & 0 & 0 & C_1 + C_3 & 0 \\ 0 & 0 & C_3 & 0 & C_2 + C_3 x_4^2 \end{pmatrix}.$$

It is clear that in the above notation $r = 2$. Let $\bar{g}_1 = g$, $\bar{g}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $\bar{g}_3 =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & x_4^2 \end{pmatrix}. \text{ We obtain that } \bar{g}_2 = \theta \otimes \theta. \text{ Consider the decomposition}$$

$$\bar{g}_3 = C_{13}g_1 + C_{23}g_2 + C_{33}\theta \otimes \theta.$$

Let $x = (\frac{\pi}{2}, 0, 0, 0, 0)$. Note that the vector $q_x = (\partial_{x_5})_x$ is light-like and satisfies $g_x(p_x, q_x) = 1$. Since $\bar{g}_{3x}(p_x, q_x) = 1$, we get $C_{23} = 1$. We change \bar{g}_3 to $\bar{g}_3 - g$. Then $C_{23} = 0$. Next,

$$\ker \bar{g}_{3x} = \text{span}\{(\partial_{x_1} - \partial_{x_4})_x, (\partial_{x_3})_x, (\partial_{x_5})_x\}, \quad \ker(g_x|_{\ker \bar{g}_{3x} \times \ker \bar{g}_{3x}}) = 0,$$

i.e. $C_{33} = 0$. We change \bar{g}_3 to $\bar{g}_3 + \theta \otimes \theta$, then $C_{33} = 1$ and

$$\bar{g}_3 = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & -\sin^2(x_1 + x_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now,

$$W = \ker \bar{g}_{2x} \cap \ker \bar{g}_{3x} = \text{span}\{(\partial_{x_1} - \partial_{x_4})_x, (\partial_{x_3})_x\}, \quad W^{\perp_g} = \text{span}\{(\partial_{x_1})_x, (\partial_{x_2})_x, (\partial_{x_3})_x\}.$$

Using Lemma 3, we find

$$E_{2x} = \text{span}\{(\partial_{x_1} - \partial_{x_4})_x, (\partial_{x_3})_x, (\partial_{x_5})_x\}.$$

Consequently, $E_{1x} = E_{2x}^\perp = \text{span}\{(\partial_{x_1})_x, (\partial_{x_2})_x\}$. Using (2) evaluated at the point x , we get

$$g_{1x} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_{2x} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, $\bar{g}_{3x} = -g_{1x} + \theta_x \otimes \theta_x$. That implies

$$g_1 = -\bar{g}_3 + \theta \otimes \theta = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & \sin^2(x_1 + x_4) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_2 = g - g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & x_4^2 \end{pmatrix}.$$

Using (7), we see that the distribution E_1 is spanned by the vector fields ∂_{x_1} , ∂_{x_2} , and the distribution E_2 is spanned by the vector fields $\partial_{x_1} - \partial_{x_4}$, ∂_{x_3} , ∂_{x_5} . We find the coordinates z_1, \dots, z_5 adopted to the decomposition $TM = E_1 \oplus E_2$ requiring that

$$\partial_{z_1} = \partial_{x_1}, \quad \partial_{z_2} = \partial_{x_2}, \quad \partial_{z_3} = \partial_{x_3}, \quad \partial_{z_4} = \partial_{x_1} - \partial_{x_4}, \quad \partial_{z_5} = \partial_{x_5},$$

and we get

$$z_1 = x_1 + x_4, \quad z_2 = x_2, \quad \dots, \quad z_5 = x_5.$$

With respect to these coordinates it holds

$$g = (dz_1)^2 + \sin^2 z_1 (dz_2)^2 + 2dz_3 dz_5 + (dz_4)^2 + z_4^2 (dz_5)^2,$$

and we obtain the initial metric.

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